EC304: Probability Theory and Stochastic Process

Module 2: Random Variables (Stochastic Variables)

1 Introduction

In real-world systems (like communication channels or sensor measurements), outcomes are often uncertain or unpredictable. To mathematically model such uncertainty, we use **Random Variables (RVs)** or **Stochastic Variables**.

A random variable (RV) is a variable that takes on different numerical values based on the outcome of a random experiment.

2 Definition

A random variable X is a function that assigns a real number to each outcome of a sample space S:

$$X : S \to \mathbb{R}$$

Example: When tossing a coin: $S = \{H, T\}$, we can define $X(H) = 1$, $X(T) = 0$.

3 Conditions for a Function to be a Random Variable

A **random variable (RV)** is a function that assigns a numerical value to each outcome of a random experiment. For a function X to be a valid random variable, it must satisfy the following three properties:

Single-Valued Function

Meaning: Each outcome of the experiment must give **exactly one value** of X. No outcome should give multiple values or be left undefined.

Mathematically:

$$\forall \omega \in S, \quad X(\omega) \in \mathbb{R}$$

Example:

- Experiment: Toss a coin once. Sample space: $S = \{H, T\}$
- Define $X(H) = 1, \quad X(T) = 0$

Each outcome has exactly one number. Valid random variable.

Invalid Case: $X(H) = \{1, 2\}$, $X(T) = 0$ Invalid because outcome H gives two values.

Measurability (Probability Condition)

Meaning: For any real number x, the set

$$\{\omega \in S : X(\omega) \le x\}$$

must be an event in the sample space so that its probability is well-defined.

Example:

- Experiment: Roll a die. Sample space: $S = \{1, 2, 3, 4, 5, 6\}$
- Define $X(\omega) = \omega$
- If $x = 3$:
$$\{\omega : X(\omega) \leq 3\} = \{1, 2, 3\}$$
Probability: $P(X \leq 3) = \frac{3}{6} = 0.5$

Measurable random variable.

Finite Probability Condition

Meaning: The random variable should take only **finite values** with probability 1.

$$P(X = +\infty) = 0, \quad P(X = -\infty) = 0$$

Example:

- Experiment: Number of heads in three coin tosses.
- Possible values: $X \in \{0, 1, 2, 3\}$

All values are finite. Valid.

Invalid Case: If $P(X = +\infty) = 0.1$, then X is not valid.

Examples: Checking Whether a Function is a Random Variable

A function $X : S \to \mathbb{R}$ is said to be a **random variable** if it satisfies the following conditions:

1. **Single-Valued Function:** Each outcome $\omega \in S$ is assigned exactly one real number $X(\omega)$.
2. **Measurability:** For every real number x, the set $\{\omega \in S : X(\omega) \leq x\}$ must be an event (i.e., its probability is well-defined).
3. **Finite Probability Condition:** The probability that X takes infinite values must be zero, i.e.,
$$P(X = \pm\infty) = 0$$

The following examples illustrate both valid and invalid cases.

Valid Case 1: Rolling a Die

Let the sample space be
$$S = \{1, 2, 3, 4, 5, 6\}.$$
Define
$$X(s) = s.$$
Here, each outcome s corresponds to a unique, finite, and measurable real number. Hence, X is a valid random variable.

Invalid Case 1: Not Single-Valued

Let $S = \{H, T\}$ represent the outcomes of a coin toss. Define

$$X(H) = \{1, 2\}, \quad X(T) = 0.$$

The outcome H maps to two values (1 and 2), which violates the single-valued property. Hence, X is **not** a random variable.
Correct Version:

$$X(H) = 1, \quad X(T) = 0.$$

Invalid Case 2: Not Measurable

Let $S = [0, 1]$ and define

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \text{ is rational,} \\ 0, & \text{if } \omega \text{ is irrational.} \end{cases}$$

Here, the set $\{\omega : X(\omega) = 1\}$ corresponds to the rational numbers in $[0, 1]$, which are not measurable in the usual probability sense. Thus, X is not a measurable random variable.
Correct Version:

$$X(\omega) = \begin{cases} 1, & \omega > 0.5, \\ 0, & \omega \leq 0.5. \end{cases}$$

In this case, the set $\{\omega : X(\omega) \leq 0.5\} = [0, 0.5]$ is measurable.

Invalid Case 3: Infinite Values with Nonzero Probability

Let $S = \{1, 2, 3, 4, 5, 6\}$ and define

$$X(s) = \begin{cases} s, & s \neq 6, \\ +\infty, & s = 6. \end{cases}$$

Since $P(X = +\infty) = P(s = 6) = \frac{1}{6} \neq 0$, the finite probability condition is violated. Therefore, X is not a valid random variable.
Correct Version: $X(s) = s$ for all $s = 1, 2, 3, 4, 5, 6$.

Invalid Case 4: Undefined Mapping

Let $S = \{1, 2, 3\}$ and define

$$X(1) = 0, \quad X(2) = 1.$$

No value is assigned to outcome 3, so X is not defined for all $\omega \in S$. Hence, X is not a valid random variable.
Correct Version:

$$X(1) = 0, \quad X(2) = 1, \quad X(3) = 2.$$

4 Types of Random Variables

(a) Discrete Random Variable

Takes on a finite or countably infinite number of distinct values.

$$p(x_i) = P(X = x_i)$$

Properties:

$$p(x_i) \geq 0, \quad \sum_i p(x_i) = 1$$

(b) Continuous Random Variable

Takes on any value in a continuous range.

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

where $f(x)$ is the **Probability Density Function (PDF)**.

$$f(x) \geq 0, \quad \int_{-\infty}^{\infty} f(x)\, dx = 1$$

Cumulative Distribution Function (CDF):

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

5 Expected Value and Variance

For discrete RV:

$$E[X] = \sum_i x_i p(x_i)$$

For continuous RV:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Variance:

$$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

6 Common Probability Distributions

Type	Distribution and Function	Parameters
Discrete	Bernoulli: $p(x) = p^x(1-p)^{1-x}$	p
Discrete	Binomial: $p(x) = \binom{n}{x} p^x (1-p)^{n-x}$	n, p
Discrete	Poisson: $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$	λ
Continuous	Uniform: $f(x) = \frac{1}{b-a}, a \leq x \leq b$	a, b
Continuous	Exponential: $f(x) = \lambda e^{-\lambda x}, x \geq 0$	λ
Continuous	Gaussian: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ, σ

6.1 Discrete Probability Distributions

Bernoulli Distribution

A Bernoulli distribution represents an experiment with only two possible outcomes: success (1) and failure (0). It has one parameter, p, which is the probability of success. This distribution is the simplest discrete model and forms the basis for many others. It is used in binary processes like a coin toss, pass/fail testing, or defective/non-defective items.

Binomial Distribution

The Binomial distribution represents the number of successes in n independent Bernoulli trials. It depends on two parameters: n (number of trials) and p (probability of success). It is commonly used when the same experiment is repeated several times, such as finding the number of heads in 10 coin tosses.

Geometric Distribution

The Geometric distribution models the number of trials required to get the first success. It has one parameter, p, the probability of success in each trial. It is used in waiting-time or reliability problems—for example, the number of attempts before the first success.

Poisson Distribution

The Poisson distribution models the number of times an event occurs in a fixed interval of time or space when events occur randomly and independently. It has one parameter, λ, representing the mean rate of occurrence. It is widely used for rare events—such as the number of phone calls per minute, accidents per day, or defects per production batch.

6.2 Continuous Probability Distributions

Continuous Uniform Distribution

A Continuous Uniform distribution assumes equal probability density across an interval $[a, b]$. It is used when all values in a certain range are equally likely, such as random number generation within a specified interval.

Normal (Gaussian) Distribution

The Normal distribution is one of the most important in statistics. It is a bell-shaped, symmetric curve defined by the mean μ and variance σ^2. Most natural phenomena such as height, weight, and IQ scores follow this distribution. It is also the basis of the Central Limit Theorem.

Exponential Distribution

The Exponential distribution represents the time between two consecutive events in a Poisson process. It depends on a rate parameter λ. It is widely used in reliability studies, modeling time until failure, or waiting-time analysis.

7 PDF and CDF for Discrete Random Variable

Example: PDF and CDF of Number of Heads in Three Coin Tosses

Consider an experiment of tossing three fair coins. Let the random variable X denote the number of heads obtained.

Sample Space

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}, \quad |S| = 8$$

$$X(\omega) = \text{Number of heads in outcome } \omega$$

Thus, X can take the values $0, 1, 2, 3$.

Probability Distribution (PMF)

Each outcome is equally likely with probability $\frac{1}{8}$.

$$P(X = x) = \begin{cases} \frac{1}{8}, & x = 0, \\ \frac{3}{8}, & x = 1, \\ \frac{3}{8}, & x = 2, \\ \frac{1}{8}, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Although X is a discrete random variable, its probability distribution can be expressed in the form of a continuous probability density function using the Dirac delta function $\delta(x)$.

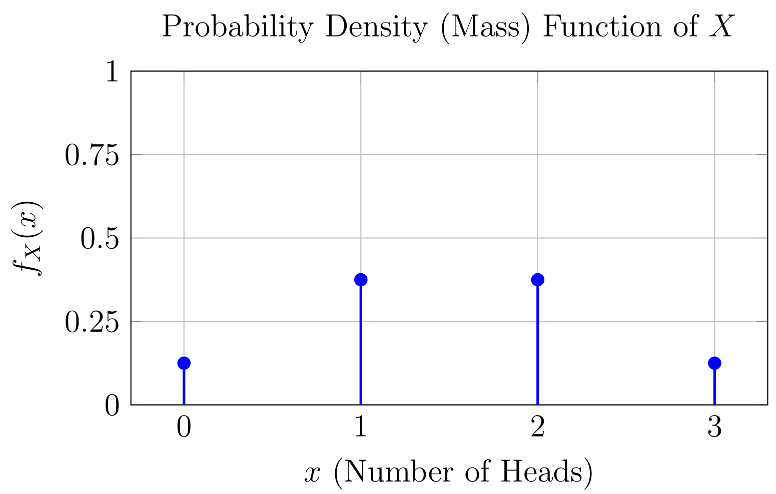
$$f_X(x) = \frac{1}{8}\,\delta(x) + \frac{3}{8}\,\delta(x-1) + \frac{3}{8}\,\delta(x-2) + \frac{1}{8}\,\delta(x-3)$$

where $\delta(x-a)$ denotes the Dirac delta function centered at $x = a$.

Verification

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Thus, the total probability equals unity, confirming that $f_X(x)$ is a valid probability density representation of X.



Cumulative Distribution Function (CDF)

$$F(x) = P(X \leq x) = \begin{cases} 0, & x < 0, \\ \frac{1}{8}, & 0 \leq x < 1, \\ \frac{1}{2}, & 1 \leq x < 2, \\ \frac{7}{8}, & 2 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$$

CDF of a Discrete Random Variable

For a random experiment where three coins are tossed, let the random variable X represent the number of heads. The possible values of X are $\{0, 1, 2, 3\}$.

The cumulative distribution function (CDF) is obtained by integrating the PMF:

$$F_X(x) = \frac{1}{8}u(x) + \frac{3}{8}u(x-1) + \frac{3}{8}u(x-2) + \frac{1}{8}u(x-3)$$

where $u(x)$ is the **unit step function** defined as:

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

—

The CDF in piecewise form is:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \leq x < 1 \\ \frac{4}{8}, & 1 \leq x < 2 \\ \frac{7}{8}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

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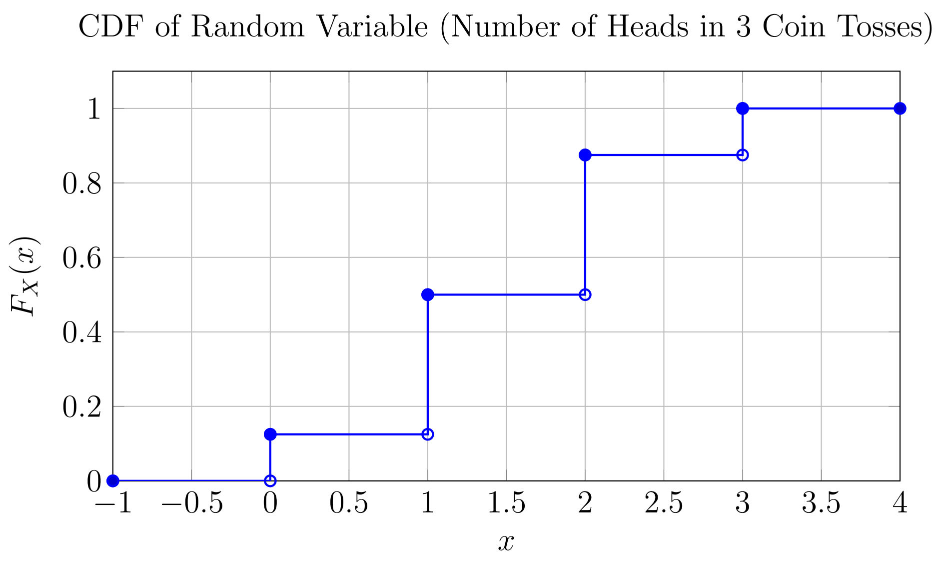


Significance of PDF and CDF Graphs

- **Most Probable Outcomes:** The highest bar in the PMF shows the most likely outcomes. Here, $X = 1$ or $X = 2$ heads are most probable.
- **Distribution Pattern:** The PMF shape shows the probability distribution pattern (symmetrical in this case).
- **Threshold Probabilities:** The CDF allows us to find probabilities for intervals. For example, $F_X(1) = 0.5$ indicates a 50% chance of getting 1 or fewer heads.

8 PDF and CDF for Continuous Random Variable

A **Continuous Random Variable** is a variable that can take any value within a given range (interval) of real numbers. Unlike discrete random variables, which take countable values (e.g., number of heads, dice outcomes), continuous variables take values from a *continuum.*

Examples:

- The voltage across a resistor.
- The temperature of a room.
- The lifetime of an electronic device.

The **Probability Density Function (PDF)** of a continuous random variable X, denoted as $f_X(x)$, describes the relative likelihood that X takes a particular value.

Properties of PDF

1. $f_X(x) \geq 0 \quad \forall x$
2. The total area under the curve is 1:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

3. The probability that X lies between two values a and b is:

$$P(a \leq X \leq b) = \int_{a}^{b} f_X(x)\, dx$$

4. $P(X = x) = 0$ for any single value x, because the probability is defined over intervals.

The **Cumulative Distribution Function (CDF)**, denoted by $F_X(x)$, gives the probability that X takes a value less than or equal to x:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\, dt$$

Properties of CDF

1. $F_X(-\infty) = 0, \quad F_X(\infty) = 1$
2. $F_X(x)$ is a non-decreasing and continuous function.
3. The PDF is the derivative of the CDF:

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Examples: Continuous PDF and CDF

Example 1: Uniform(0,1) — simple interval probability

Let $X \sim U(0,1)$ with

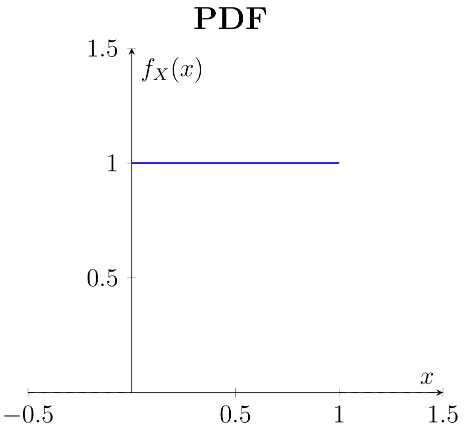
$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

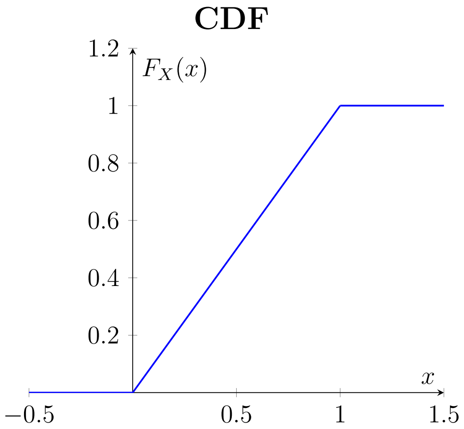
(a) Find the CDF $F_X(x)$.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, & x < 0, \\ x, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

(b) Compute $P(0.2 < X < 0.7)$.

$$P(0.2 < X < 0.7) = \int_{0.2}^{0.7} 1\,dx = 0.7 - 0.2 = 0.5.$$





Example 2: Exponential distribution (rate λ)

Let X have PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0, \\ 0, & x < 0, \end{cases} \qquad \lambda > 0.$$

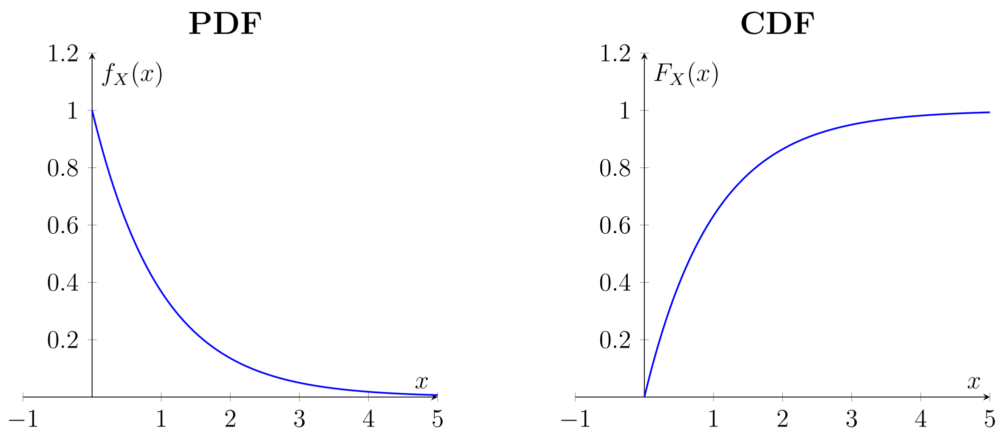
(a) CDF $F_X(x)$.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \ge 0. \end{cases}$$

(b) Tail probability $P(X > a)$.

$$P(X > a) = 1 - F_X(a) = e^{-\lambda a}.$$

(c) Mean and variance. $\mathbb{E}[X] = 1/\lambda, \quad \mathrm{Var}(X) = 1/\lambda^2.$



Example 3: PDF $f_X(x) = 2x$ on $[0, 1]$ (Beta(2,1))

Let

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) CDF $F_X(x)$. For $0 \le x \le 1$,

$$F_X(x) = \int_0^x 2t\,dt = [t^2]_0^x = x^2.$$

So

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x^2, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

(b) Probability $P(0.5 < X < 1)$.

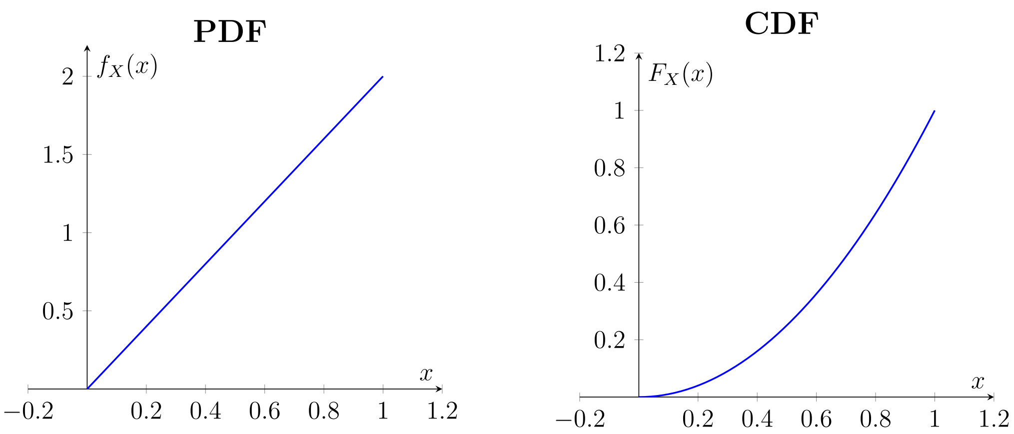
$$P(0.5 < X < 1) = F_X(1) - F_X(0.5) = 1 - (0.5)^2 = 1 - 0.25 = 0.75.$$

(c) Mean and variance.

$$\mathbb{E}[X] = \int_0^1 x \cdot 2x\,dx = 2\int_0^1 x^2\,dx = 2 \cdot \frac{1}{3} = \frac{2}{3}.$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot 2x\,dx = 2\int_0^1 x^3\,dx = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18} \approx 0.05556.$$



Example 4: Normal $N(0, 1)$ — using the CDF

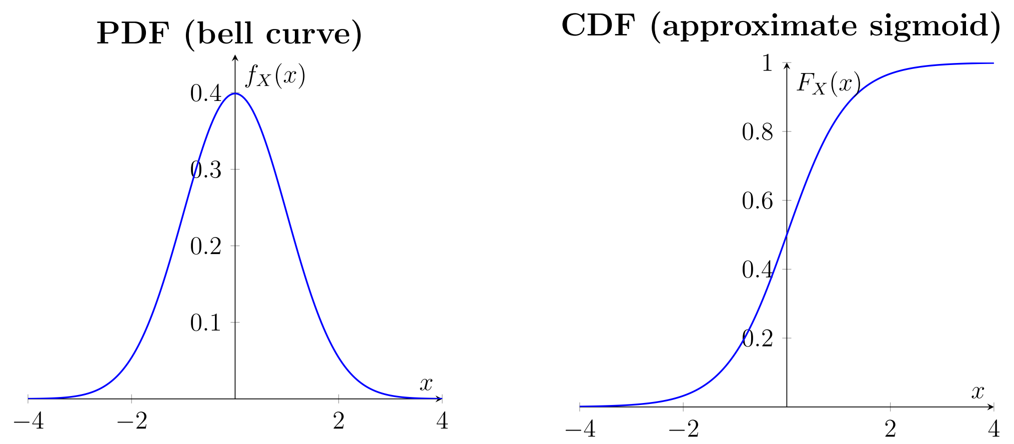
Let $X \sim N(0, 1)$ with CDF $\Phi(x)$ (standard normal CDF).
Compute $P(-1 < X < 2)$.

$$P(-1 < X < 2) = \Phi(2) - \Phi(-1).$$

Using standard values (or a calculator / table): $\Phi(2) \approx 0.97725$, $\Phi(-1) \approx 0.15866$. Thus

$$P(-1 < X < 2) \approx 0.97725 - 0.15866 = 0.81859 \ (\approx 81.86\%).$$

Note: For the standard normal, you normally use tables or software to evaluate Φ.



Example 5: Triangular PDF on $[0, 2]$ with peak at 1

Define

$$f_X(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(Verify normalization: $\int_0^1 x\,dx + \int_1^2 (2 - x)\,dx = \frac{1}{2} + \frac{1}{2} = 1$.)

(a) CDF. For $0 \le x \le 1$,

$$F_X(x) = \int_0^x t\,dt = \frac{x^2}{2}.$$

For $1 < x \le 2$,

$$F_X(x) = F_X(1) + \int_1^x (2-t)\,dt = \frac{1}{2} + \left[2t - \frac{t^2}{2}\right]_1^x = -1 + 2x - \frac{x^2}{2}.$$

So

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{2}, & 0 \le x \le 1, \\ -1 + 2x - \frac{x^2}{2}, & 1 < x \le 2, \\ 1, & x > 2. \end{cases}$$

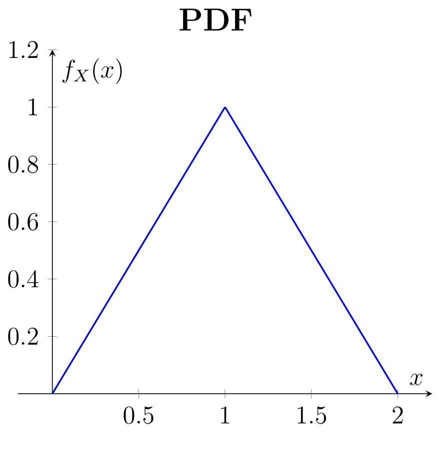
(b) Compute $P(X \le 1.5)$. For $x = 1.5$ (in $1 < x \le 2$):

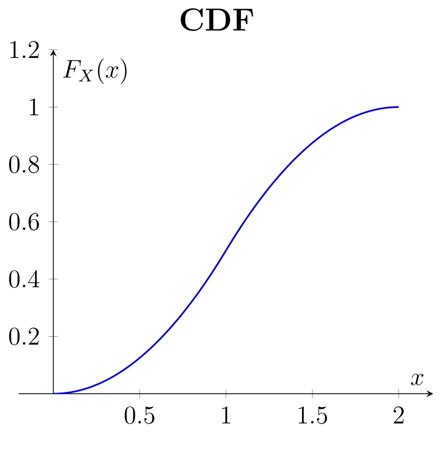
$$F_X(1.5) = -1 + 2(1.5) - \frac{(1.5)^2}{2} = -1 + 3 - \frac{2.25}{2} = 2 - 1.125 = 0.875.$$

So $P(X \le 1.5) = 0.875$.

(c) Mean and variance (symmetric triangle). Because the PDF is symmetric about $x = 1$, $\mathbb{E}[X] = 1$. The variance of a symmetric triangular distribution on $[a, b]$ with peak at the midpoint is $(b-a)^2/24$. Here $b - a = 2$, so

$$\text{Var}(X) = \frac{4}{24} = \frac{1}{6} \approx 0.1667.$$





Example 6: Recover PDF from given CDF

Suppose

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-x}, & x \ge 0. \end{cases}$$

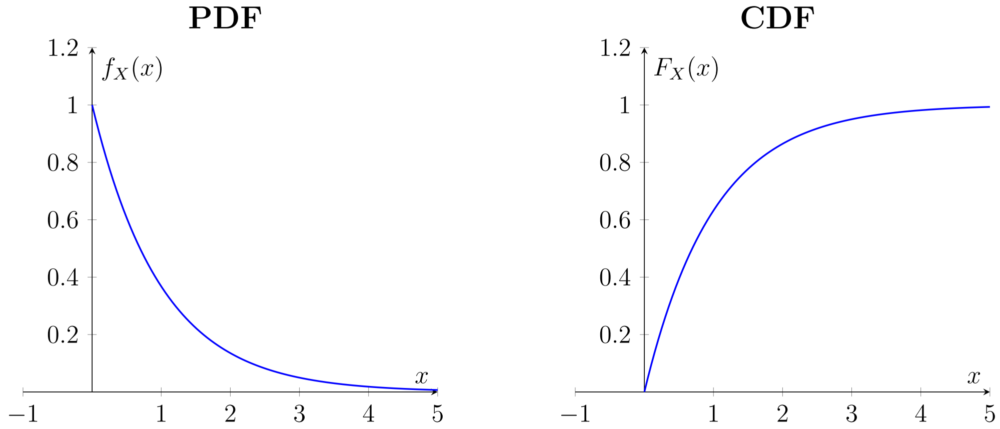
Find the PDF.

Differentiate the CDF where it is differentiable:

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 0, & x < 0, \\ e^{-x}, & x \geq 0, \end{cases}$$

which is the exponential PDF with $\lambda = 1$.

Given $F_X(x) = 1 - e^{-x}, x \geq 0$, the PDF is $f_X(x) = e^{-x}$.



Example 7:

A discrete random variable X takes values

$$x_i = -1, 0, 1, 2, 3, 4, 5, 6, 7$$

with

$$f_X(x_i) = K,\ 2K,\ 3K,\ K,\ 4K,\ 3K,\ 2K,\ 4K,\ K$$

respectively.

(i) Find K

Sum of probabilities = 1:

$$(1+2+3+1+4+3+2+4+1)K = 21K = 1 \quad \Rightarrow \quad K = \frac{1}{21}.$$

Numerical PMF

Substitute $K = \frac{1}{21}$.

x	-1	0	1	2	3	4	5	6	7
$f_X(x)$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{1}{21}$	$\frac{4}{21}$	$\frac{3}{21}$	$\frac{2}{21}$	$\frac{4}{21}$	$\frac{1}{21}$
Decimal	0.0476	0.0952	0.1429	0.0476	0.1905	0.1429	0.0952	0.1905	0.0476

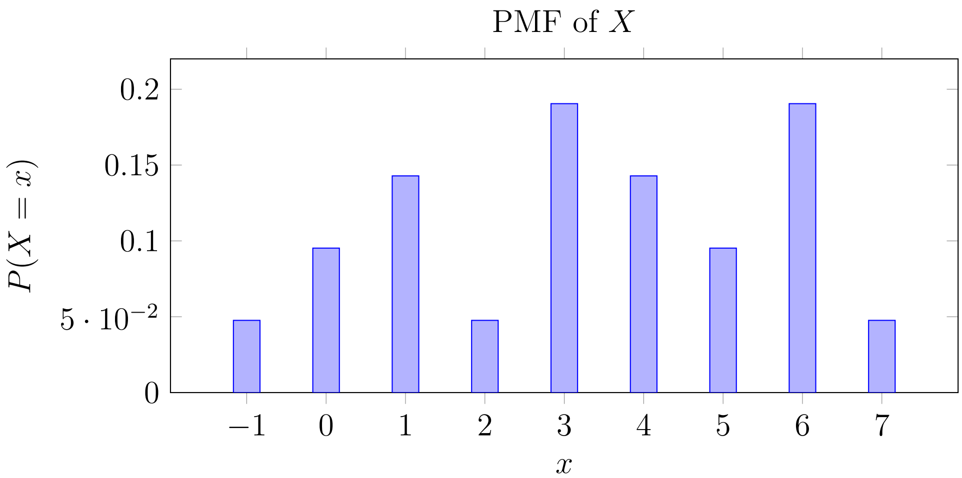
Requested probabilities (using $K = \frac{1}{21}$)

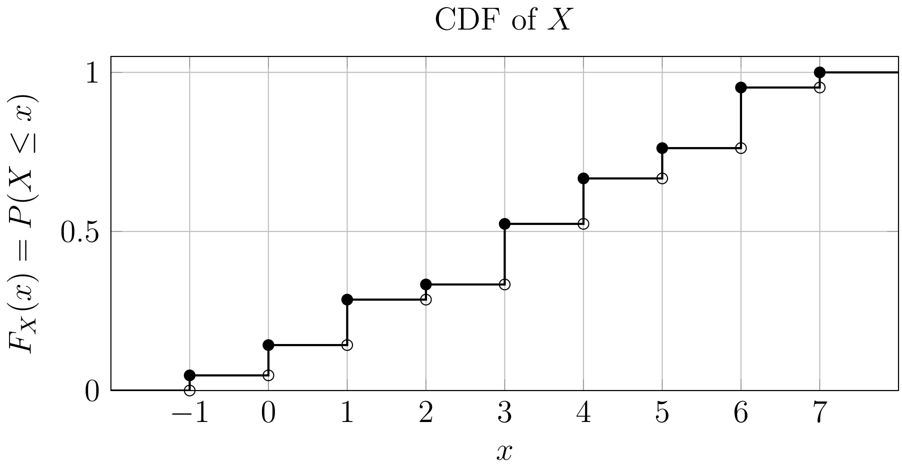
(ii) $P(X > 2) = P(X \in \{3,4,5,6,7\}) = (4+3+2+4+1)K = 14K = \frac{14}{21} = \frac{2}{3} \approx 0.6667.$

(iii) $P(X > 4) = P(X \in \{5,6,7\}) = (2+4+1)K = 7K = \frac{7}{21} = \frac{1}{3} \approx 0.3333.$

(iv) $P(1 < X \leq 4) = P(X \in \{2,3,4\}) = (1+4+3)K = 8K = \frac{8}{21} \approx 0.3810.$

PMF and CDF Plots





Example 8: Gaussian (Normal) Noise in Sensors

In real-world measurements, sensors rarely produce perfectly accurate readings. Small random fluctuations known as **noise** are always present due to environmental, electrical, or internal variations in the sensor. When these variations follow a *bell-shaped probability distribution*, they are modeled as **Gaussian (Normal) Noise**.

Mathematical Representation

A Gaussian random variable X is defined by two parameters — mean (μ) and standard deviation (σ) — and its Probability Density Function (PDF) is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where:

- μ : Mean (average value, around which noise is centered)
- σ : Standard deviation (measure of spread or noise strength)

Interpretation

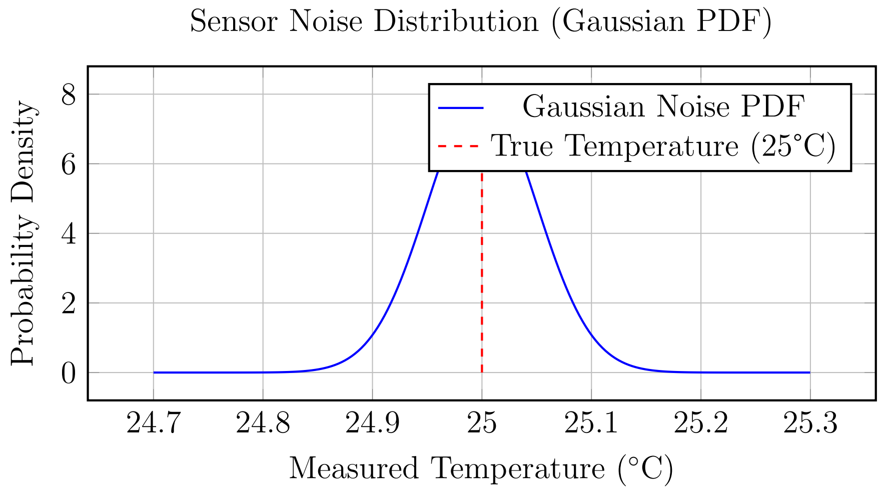
- Most of the measured values are close to the true value.
- The probability of large deviations decreases exponentially.
- If σ is small, the sensor is more precise; if large, it is noisier.

Suppose a temperature sensor is measuring the true value of $25°C$. Due to random noise, the sensor readings fluctuate slightly. This noise can be modeled as a Gaussian random variable with $\mu = 25$ and $\sigma = 0.05$. Hence, the measured readings are:

$$X = 25 + N(0, 0.05^2)$$

This means each reading deviates slightly around 25 with most values falling between 24.9 and 25.1.

Graphical Representation



Explanation of the Graph

The curve is bell-shaped and centered at $25°C$. It shows that:

- Measurements near $25°C$ are most frequent (high probability).
- Readings far from $25°C$ are rare (low probability).
- The spread (width) depends on σ — smaller σ means a narrower, sharper curve.