EC304: Probability Theory and Stochastic Process

Module 3: Statistical Parameters of Random Variables

1 Introduction

Statistical parameters describe the characteristics of a random variable's probability distribution. The most important parameters are:

- Mean or Expected Value
- Variance and Standard Deviation
- Higher-order Moments
- Covariance and Correlation

These parameters help quantify the central tendency, dispersion and shape of a distribution.

2 Mean or Expected Value

2.1 Discrete Random Variable

$$E[X] = \sum_x x\, P(X = x)$$

2.2 Continuous Random Variable

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Interpretation: The expected value is the center of mass of the probability distribution.

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Worked Example 1 (Discrete)

A random variable X takes values $\{1, 2, 3\}$ with probabilities $\{0.2, 0.5, 0.3\}$.

$$E[X] = 1(0.2) + 2(0.5) + 3(0.3) = 2.1$$

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Worked Example 2 (Continuous)

Let

$$f(x) = 3x^2, \quad 0 < x < 1$$

$$E[X] = \int_0^1 3x^2 \cdot x\, dx = \frac{3}{5} = 0.6$$

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3 Variance and Standard Deviation

Variance measures how spread out the distribution is:

$$\text{Var}(X) = E[(X - \mu)^2]$$

Alternate form:

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

Standard deviation:

$$\sigma_X = \sqrt{\text{Var}(X)}$$

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Worked Example 3

Using Example 1:

$$E[X^2] = 1^2(0.2) + 2^2(0.5) + 3^2(0.3) = 4.9$$

$$\text{Var}(X) = 4.9 - (2.1)^2 = 0.49$$

$$\sigma_X = 0.7$$

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Worked Example 4 (Continuous)

Using Example 2:

$$E[X^2] = \int_0^1 3x^2 \cdot x^2 dx = \frac{3}{6} = \frac{1}{2}$$

$$\text{Var}(X) = \frac{1}{2} - (0.6)^2 = 0.14$$

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4 Higher-Order Moments

- **n-th Moment About Origin:** $E[X^n]$
- **Central Moments:** $E[(X - \mu)^n]$

Example 5: For $X = \{1, 2, 3\}$, equally likely:

$$E[X^3] = \frac{1 + 8 + 27}{3} = 12$$

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5 Additional Worked Numerical Examples

Example 7: Variance of a Biased Coin

Let

$$X = \begin{cases} 1 & \text{with prob } 0.7 \\ 0 & \text{with prob } 0.3 \end{cases}$$

$$E[X] = 0.7, \quad E[X^2] = 0.7$$

$$\text{Var}(X) = 0.7 - (0.7)^2 = 0.21$$

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Example 8: Uniform Distribution

For $X \sim U(a, b)$:

$$E[X] = \frac{a+b}{2}, \qquad \text{Var}(X) = \frac{(b-a)^2}{12}$$

Let $X \sim U(2, 6)$:

$$E[X] = 4, \quad \text{Var}(X) = \frac{(6-2)^2}{12} = \frac{16}{12} = 1.33$$

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6 Exercise Problems

Exercise 1

A random variable X has PMF:

$$P(X = x) = \begin{cases} kx^2, & x = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

Find:

- (a) k
- (b) $E[X]$
- (c) $E[X^2]$
- (d) $\text{Var}(X)$

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Exercise 2

For a continuous PDF:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

compute:

- (a) $E[X]$
- (b) $\text{Var}(X)$

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Exercise 3

Given

$$f(x) = 2x,\ 0 < x < 1,$$

find:

- (a) Median
- (b) Mode
- (c) $P(0.3 < X < 0.7)$

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Exercise 4

Random variables X and Y have:

$$E[X] = 3, \quad E[Y] = 5, \quad \mathrm{Var}(X) = 4, \quad \mathrm{Var}(Y) = 9, \quad \mathrm{Cov}(X, Y) = 3$$

Compute:

$$\mathrm{Var}(2X - 3Y + 5)$$

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7 Summary Table

Parameter	**Formula**	**Meaning**
Mean	$E[X]$	Central value
Variance	$E[(X-\mu)^2]$	Spread
Std. Dev.	$\sqrt{\mathrm{Var}(X)}$	Spread in same units
n-th Moment	$E[X^n]$	Shape
Covariance	$E[(X-\mu_X)(Y-\mu_Y)]$	Linear dependence
Correlation	ρ	Normalized dependence