EC304: Probability Theory and Stochastic Process

Module 6: Probability Generating Function (PGF)

1. Introduction

The **Probability Generating Function (PGF)** is a powerful tool in probability theory used to study **discrete random variables**, especially those that take non-negative integer values (such as counts of events).

It simplifies calculations involving probabilities, expectations, and variances and is useful in the analysis of **queueing systems, communication errors, and reliability models**.

2. Definition

Let X be a discrete random variable that takes non-negative integer values $0, 1, 2, \dots$ with probability mass function (PMF) $P(X = k) = p_k$.

Then the **probability generating function** (PGF) of X is defined as:

$$G_X(s) = E[s^X] = \sum_{k=0}^{\infty} p_k s^k, \quad \text{for } |s| \leq 1$$

Note: The PGF uniquely determines the probability distribution of X.

3. Properties of PGF

1. $G_X(1) = \sum_{k=0}^{\infty} p_k = 1$
2. $G_X(0) = P(X = 0)$
3. The n^{th} derivative of $G_X(s)$ gives:

$$G_X^{(n)}(s) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1) p_k s^{k-n}$$

4. **Mean (Expectation):**

$$E[X] = G'_X(1)$$

5. **Variance:**

$$Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2$$

6. If $X_1, X_2, \dots, X_n$ are independent and identically distributed (i.i.d.) random variables, and $S_n = X_1 + X_2 + \cdots + X_n$, then:

$$G_{S_n}(s) = [G_X(s)]^n$$

4. Examples with Numerical Illustrations

Example 1: Bernoulli Distribution

A **Bernoulli Distribution** is the simplest discrete probability distribution. It models a random experiment that has only two possible outcomes:

- **Success (1)** with probability p
- **Failure (0)** with probability $1 - p$

Let X be a random variable defined as:

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1 - p \end{cases}$$

Then X is said to follow a **Bernoulli distribution** with parameter p, denoted as:

$$X \sim \text{Bernoulli}(p), \quad 0 \leq p \leq 1$$

Probability Mass Function (PMF)

$$P(X = x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$

Alternatively, the PMF can be written compactly as:

$$P(X = x) = p^x (1 - p)^{1-x}, \quad x = 0, 1$$

Probability Generating Function (PGF)

The probability generating function (PGF) of a Bernoulli random variable X is:

$$G_X(s) = (1 - p) + ps$$

From this,

$$G'_X(1) = p = E[X], \quad Var(X) = p(1 - p)$$

Numerical Example:1 If $p = 0.4$, then

$$G_X(s) = 0.6 + 0.4s$$

$$E[X] = 0.4, \quad Var(X) = 0.4(0.6) = 0.24$$

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Numerical Example:2 Suppose a mobile packet transmission succeeds with probability $p = 0.8$ and fails with probability 0.2. Let $X = 1$ if the transmission is successful, and $X = 0$ otherwise.

$$P(X = 1) = 0.8, \quad P(X = 0) = 0.2$$
$$E[X] = 0.8, \quad Var(X) = 0.8(0.2) = 0.16$$
$$G_X(s) = 0.2 + 0.8s$$

Example 2: Binomial Distribution

Let $X \sim \text{Binomial}(n, p)$, then:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$G_X(s) = [(1 - p) + ps]^n$$

Mean: $E[X] = np$
Variance: $Var(X) = np(1 - p)$

Numerical Example: Let $n = 3$, $p = 0.5$

$$G_X(s) = (0.5 + 0.5s)^3 = 0.125(1 + s)^3$$

$$E[X] = 3(0.5) = 1.5, \quad Var(X) = 3(0.5)(0.5) = 0.75$$

Verification from PMF:

$$P(0) = 0.125,\ P(1) = 0.375,\ P(2) = 0.375,\ P(3) = 0.125$$

$$G_X(s) = 0.125 + 0.375s + 0.375s^2 + 0.125s^3$$
$$G'_X(s) = 0.375 + 0.75s + 0.375s^2, \quad G'_X(1) = 1.5$$

Confirmed: $E[X] = 1.5$

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Example 3: Poisson Distribution

Let $X \sim \text{Poisson}(\lambda)$, then:

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$

$$G_X(s) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{\lambda(s-1)}$$

Mean: $E[X] = \lambda$
Variance: $Var(X) = \lambda$

Numerical Example: If $\lambda = 2$, then

$$G_X(s) = e^{2(s-1)}$$
$$G'_X(s) = 2e^{2(s-1)}, \quad G'_X(1) = 2$$
$$G''_X(s) = 4e^{2(s-1)}, \quad G''_X(1) = 4$$
$$Var(X) = 4 + 2 - 2^2 = 2$$

Hence, $E[X] = 2$, $Var(X) = 2$.

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Example 4: Geometric Distribution

Let X follow a Geometric(p) distribution (count of trials up to first success):

$$P(X = k) = (1-p)^{k-1}p, \quad k = 1, 2, 3, \ldots$$

Then,

$$G_X(s) = \frac{ps}{1-(1-p)s}, \quad |s| < \frac{1}{1-p}$$

Mean: $E[X] = \frac{1}{p}$
Variance: $Var(X) = \frac{1-p}{p^2}$

Numerical Example: Let $p = 0.25$

$$G_X(s) = \frac{0.25s}{1-0.75s}$$
$$E[X] = \frac{1}{0.25} = 4, \quad Var(X) = \frac{0.75}{(0.25)^2} = 12$$

Thus, on average 4 trials are needed for one success.

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Example 5: Custom PMF

A random variable X has PMF

$$P(X = 0) = 0.2, \; P(X = 1) = 0.5, \; P(X = 2) = 0.3$$

Find PGF, mean, and variance.

Solution:

$$G_X(s) = 0.2 + 0.5s + 0.3s^2$$
$$G'_X(s) = 0.5 + 0.6s, \quad G''_X(s) = 0.6$$
$$E[X] = G'_X(1) = 1.1$$
$$Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2 = 0.6 + 1.1 - 1.21 = 0.49$$

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Applications of PGF

- **Communication Systems:** To model packet arrival rates or error counts in digital communication.
- **Queueing Theory:** To analyze number of customers in a system or delay distribution.
- **Reliability Engineering:** For modeling the number of failures in a given time.
- **Branching Processes:** To study population growth and spreading of messages in networks.

Summary

Distribution	**PGF** $G_X(s)$	**Mean**	**Variance**
Bernoulli(p)	$(1-p)+ps$	p	$p(1-p)$
Binomial(n,p)	$[(1-p)+ps]^n$	np	$np(1-p)$
Poisson(λ)	$e^{\lambda(s-1)}$	λ	λ
Geometric(p)	$\frac{ps}{1-(1-p)s}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$