EC304: Probability Theory and Stochastic Process

Module 7: Stochastic (Random) Process

1 Introduction

A **Random Process** (or **Stochastic Process**) is a collection of random variables $\{X(t)\}$, where each random variable represents the value of a quantity at a particular time t.

$$X(t) : t \in T$$

Here:

- t — the time index (can be continuous or discrete)
- $X(t)$ — random variable at time t
- T — index set (usually time domain)

Thus, a random process describes how a random variable evolves over time.

Example: Thermal noise voltage in a resistor, $n(t)$, is random in time. Each time you measure $n(t)$, you get a different waveform (sample function).

$$\{n(t,\omega_1), n(t,\omega_2), n(t,\omega_3), \dots\}$$

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2 Classification of Random Processes

Random (Stochastic) processes can be classified in several ways depending on their time index, amplitude, stationarity, and statistical properties.

2.1 Non-Deterministic Random Process

A process whose future values cannot be predicted exactly even if the past behavior is known.

Continuous Random Process: Both time and amplitude are continuous. Example: Thermal noise voltage.

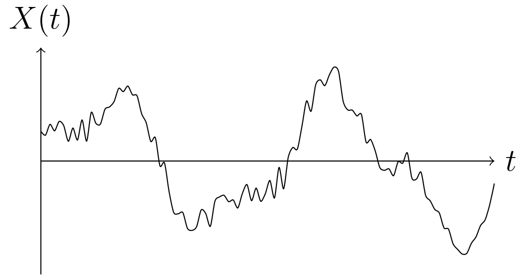

Figure 1: Continuous Random Process (e.g., Thermal noise)

Discrete Random Process: Time is discrete but amplitude is continuous. Example: Sampled noise at discrete time intervals.

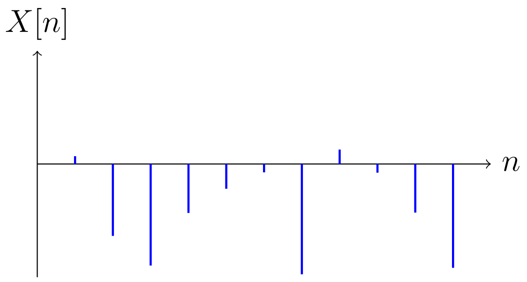

Figure 2: Discrete Random Process (e.g., Sampled noise sequence)

2.2 Deterministic Random Process

If the random parameters are known, the process becomes predictable or deterministic. Example:

$$X(t) = A\sin(\omega t + \Theta)$$

where Θ is a random variable uniformly distributed in $(0, 2\pi)$. Once Θ is known, $X(t)$ is completely determined.

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2.3 Stationary Random Process

A random process whose statistical properties do not change with time.

Strict Sense Stationary (SSS): All joint probability distributions are invariant with respect to time shift.

$$F_{X(t_1),X(t_2),...,X(t_n)}(x_1, ..., x_n) = F_{X(t_1+\tau),X(t_2+\tau),...,X(t_n+\tau)}(x_1, ..., x_n)$$

Wide Sense Stationary (WSS): Only mean and autocorrelation are time-invariant:

$$E[X(t)] = \text{constant}, \quad R_X(t_1, t_2) = R_X(t_2 - t_1)$$

Example: $X(t) = A\cos(\omega t + \Theta)$, $\Theta \sim U(0, 2\pi)$

$$E[X(t)] = 0, \quad R_X(\tau) = \frac{A^2}{2}\cos(\omega\tau)$$

Hence, $X(t)$ is WSS.

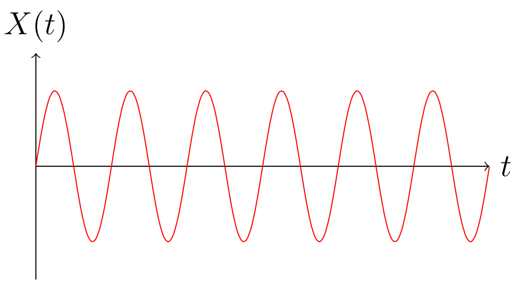

Figure 3: Stationary Random Process (constant mean and variance)

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2.4 Non-Stationary Random Process

A random process whose statistical parameters (mean or variance) vary with time. Example: Speech signal or fading channel envelope.

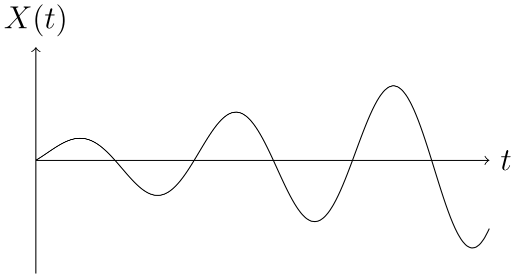

Figure 4: Non-Stationary Random Process (mean or variance changes over time)

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2.5 Ergodic Random Process

A process where time averages and ensemble averages are identical:

$$\overline{X(t)} = E[X(t)]$$

Example: Thermal noise — observing one long realization gives same statistics as ensemble of realizations.

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3 Statistical Parameters of Random Process

3.1 Mean

It is the expected value of the process at time t.

$$\mu_X(t) = E[X(t)]$$

Example:

Let

$$X(t) = A\cos(\omega t + \Theta)$$

We assume A and ω are constants and Θ is the only random variable with pdf

$$f_\Theta(\theta) = \frac{1}{2\pi}, \quad 0 \le \theta < 2\pi.$$

The mean is

$$E[X(t)] = A\,E[\cos(\omega t + \Theta)] = A\int_0^{2\pi} \cos(\omega t + \theta)\,\frac{d\theta}{2\pi}.$$

With substitution $u = \omega t + \theta$ the integral becomes

$$E[\cos(\omega t + \Theta)] = \frac{1}{2\pi}\int_{\omega t}^{\omega t + 2\pi} \cos u\,du = \frac{1}{2\pi}\big[\sin u\big]_{\omega t}^{\omega t + 2\pi} = 0.$$

Hence

$$\boxed{\mu_X(t) = E[X(t)] = 0.}$$

3.2 Autocorrelation

Autocorrelation measures how similar a signal is to a time-shifted version of itself. For a random process $X(t)$, the autocorrelation function is

$$R_X(\tau) = E[X(t)\,X(t+\tau)].$$

The autocorrelation indicates the periodicity and frequency components in the signal

Example:

Let

$$X(t) = A\cos(\omega t + \Theta)$$

The autocorrelation will be,

$$R_X(t, t+\tau) = E\big[A\cos(\omega t+\Theta)\ A\cos(\omega(t+\tau)+\Theta)\big] = A^2\, E\big[\cos(\omega t+\Theta)\cos(\omega t+\omega\tau+\Theta)\big].$$

Use the trigonometric identity

$$\cos\alpha\cos\beta = \tfrac{1}{2}\big[\cos(\alpha-\beta) + \cos(\alpha+\beta)\big],$$

with $\alpha = \omega t + \Theta,\ \beta = \omega t + \omega\tau + \Theta$. Then

$$\cos(\alpha-\beta) = \cos(-\omega\tau) = \cos(\omega\tau),$$

and

$$\cos(\alpha+\beta) = \cos(2\omega t + \omega\tau + 2\Theta).$$

Therefore

$$\begin{aligned} R_X(t, t+\tau) &= A^2 \cdot \frac{1}{2}\, E\big[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\Theta)\big] \\ &= \frac{A^2}{2}\cos(\omega\tau) + \frac{A^2}{2} E\big[\cos(2\omega t + \omega\tau + 2\Theta)\big]. \end{aligned}$$

The remaining expectation is over Θ uniform on $[0, 2\pi)$:

$$E[\cos(2\omega t + \omega\tau + 2\Theta)] = \int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta)\,\frac{d\theta}{2\pi}.$$

The integral of cosine over any interval of length 2π is zero, so the expectation is zero. Hence

$$E[\cos(2\omega t + \omega\tau + 2\Theta)] = 0.$$

Thus the autocorrelation reduces to

$$\boxed{R_X(t, t+\tau) = \frac{A^2}{2}\cos(\omega\tau).}$$

3.3 Autocovariance

Autocovariance removes the effect of mean and measures the similarity between the fluctuations of the signal:

$$C_X(\tau) = E[(X(t)-\mu)(X(t+\tau)-\mu)],$$

where $\mu = E[X(t)]$.

Note:

- Shows correlation only in the varying (AC) part of the signal.
- Useful when the signal has a large DC offset.

Example Let

$$X(t) = 3 + \sin(2\pi t),$$

mean $\mu = 3$. Autocovariance removes the constant term and highlights only the sinusoidal variation.

Consider the random process

$$X(t) = A\cos(\omega t + \Theta),$$

where A and ω are constants and Θ is uniformly distributed over $(0, 2\pi)$.

The autocovariance function is defined as

$$C_X(t, t+\tau) = E[(X(t) - \mu)(X(t+\tau) - \mu)], \quad \text{where } \mu = E[X(t)].$$

Since $E[X(t)] = 0$, we get

$$C_X(t, t+\tau) = R_X(t, t+\tau).$$

Thus,

$$\boxed{C_X(t, t+\tau) = \frac{A^2}{2}\cos(\omega\tau).}$$

If the Amplitude A is Random

If A is a random variable independent of Θ, with finite second moment,

$$R_X(\tau) = \frac{E[A^2]}{2}\cos(\omega\tau),$$

and the mean still satisfies $E[X(t)] = 0$. Therefore,

$$\boxed{C_X(\tau) = \frac{E[A^2]}{2}\cos(\omega\tau).}$$

This confirms that the autocovariance has the same form as the autocorrelation for this zero-mean random process.

Wide-Sense Stationarity (WSS)

A process is WSS if (i) the mean is constant and (ii) the autocorrelation depends only on the time difference τ. We have shown:

$$E[X(t)] = 0 \quad \text{(constant)}.$$

and

$$R_X(t, t+\tau) = \frac{A^2}{2}\cos(\omega\tau),$$

which depends only on τ, not on the absolute time t. Therefore the process is *wide-sense stationary* (WSS).

Average Power

The average power (for a WSS zero-mean process) is $P_X = R_X(0)$. From above,

$$P_X = R_X(0) = \frac{A^2}{2}\cos(0) = \frac{A^2}{2}.$$

3.4 Cross-Correlation

Cross-correlation measures similarity between two signals as one is shifted in time:

$$R_{XY}(\tau) = E[X(t)\,Y(t+\tau)].$$

Note:

- Identifies time delay between two measurements.
- Detects synchronization between sensors or systems.
- Used in radar, sonar, speech processing and communication systems.

Example: Let

$$X(t) = \sin(2\pi t), \qquad Y(t) = \sin(2\pi(t-0.2)).$$

Using the time-average definition of cross-correlation for deterministic signals:

$$R_{XY}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} X(t)\,Y(t+\tau)\,dt.$$

Compute the integrand:

$$Y(t+\tau) = \sin\big(2\pi(t+\tau-0.2)\big) = \sin\big(2\pi t + 2\pi(\tau-0.2)\big).$$

So

$$X(t)Y(t+\tau) = \sin(2\pi t)\,\sin\big(2\pi t + 2\pi(\tau-0.2)\big).$$

Use the identity $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$ with $\alpha = 2\pi t,\ \beta = 2\pi t + 2\pi(\tau-0.2)$. Then

$$\begin{aligned}
X(t)Y(t+\tau) &= \tfrac{1}{2}\Big[\cos\big(2\pi t - (2\pi t + 2\pi(\tau-0.2))\big) - \cos\big(2\pi t + (2\pi t + 2\pi(\tau-0.2))\big)\Big]\\
&= \tfrac{1}{2}\Big[\cos\big(-2\pi(\tau-0.2)\big) - \cos\big(4\pi t + 2\pi(\tau-0.2)\big)\Big]\\
&= \tfrac{1}{2}\Big[\cos\big(2\pi(\tau-0.2)\big) - \cos\big(4\pi t + 2\pi(\tau-0.2)\big)\Big].
\end{aligned}$$

Now average over t. The first term is independent of t; the second term $\cos(4\pi t + 2\pi(\tau-0.2))$ has zero time-average (it integrates to zero over any integer number of periods). Therefore

$$\begin{aligned}
R_{XY}(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\tfrac{1}{2}\cos\big(2\pi(\tau-0.2)\big)\,dt - \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\tfrac{1}{2}\cos(4\pi t + 2\pi(\tau-0.2))\,dt\\
&= \tfrac{1}{2}\cos\big(2\pi(\tau-0.2)\big) - 0.
\end{aligned}$$

$$\boxed{R_{XY}(\tau) = \tfrac{1}{2}\cos\big(2\pi(\tau-0.2)\big)}$$

Remarks:

- At $\tau = 0$: $R_{XY}(0) = \frac{1}{2}\cos(-0.4\pi) = \frac{1}{2}\cos(0.4\pi) \approx 0.1545$.
- At $\tau = 0.2$: $R_{XY}(0.2) = \frac{1}{2}\cos(0) = \frac{1}{2}$ (maximum correlation — signals aligned).
- This result is consistent with the fact that $Y(t)$ is $X(t)$ shifted by 0.2 seconds.

Here $Y(t)$ is a delayed version of $X(t)$ by 0.2 seconds. Cross-correlation will attain its maximum at $\tau = 0.2$, revealing the delay.

3.5 Power Spectral Density (PSD)

The Power Spectral Density (PSD) shows how the power of a signal is distributed across different frequencies.

Note: PSD tells us,

- Which frequencies contain most of the signal power.
- Whether the signal is dominated by low or high frequency components.
- How noise or useful signals are spread in frequency.
- Useful for design of filters, spectrum allocation in communication systems

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau}\, d\tau.$$

For a WSS process, the PSD is the Fourier transform of the autocorrelation:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau.$$

With $R_X(\tau) = \dfrac{A^2}{2}\cos(\omega\tau)$ and using $\cos(\omega\tau) = \frac{1}{2}\big(e^{j\omega\tau} + e^{-j\omega\tau}\big)$,

$$R_X(\tau) = \frac{A^2}{4}\, e^{j\omega\tau} + \frac{A^2}{4}\, e^{-j\omega\tau}.$$

Taking Fourier transform term-wise gives impulses at frequencies $f = \pm f_0$ where $f_0 = \dfrac{\omega}{2\pi}$:

$$\begin{aligned} S_X(f) &= \frac{A^2}{4}\int_{-\infty}^{\infty} e^{j\omega\tau}e^{-j2\pi f\tau}\, d\tau + \frac{A^2}{4}\int_{-\infty}^{\infty} e^{-j\omega\tau}e^{-j2\pi f\tau}\, d\tau \\ &= \frac{A^2}{4}\int_{-\infty}^{\infty} e^{-j2\pi(f-\frac{\omega}{2\pi})\tau}\, d\tau + \frac{A^2}{4}\int_{-\infty}^{\infty} e^{-j2\pi(f+\frac{\omega}{2\pi})\tau}\, d\tau \\ &= \frac{A^2}{4}\,\delta\Big(f - \frac{\omega}{2\pi}\Big) + \frac{A^2}{4}\,\delta\Big(f + \frac{\omega}{2\pi}\Big). \end{aligned}$$

Hence

$$\boxed{S_X(f) = \frac{A^2}{4}\Big[\delta\big(f - f_0\big) + \delta\big(f + f_0\big)\Big], \qquad f_0 = \frac{\omega}{2\pi}.}$$

This PSD is consistent with a deterministic sinusoid of amplitude A (which produces impulses at $\pm f_0$) but note the factor 1/4 arises because we are using cosine decomposition and ensemble averaging; the total average power (area under PSD) equals $A^2/2$ as found earlier:

$$\int_{-\infty}^{\infty} S_X(f)\, df = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2} = P_X.$$

Summary

- **Autocorrelation:** Similarity of signal with itself; reveals periodicity.
- **Autocovariance:** Autocorrelation of fluctuations after removing mean.
- **Cross-correlation:** Similarity between two signals; reveals time delay.
- **PSD:** Frequency-domain representation of how signal power is distributed.