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# EC304: Probability Theory and Stochastic Process

## Module 1: Introduction

### 1 Basics and Terminologies

#### 1.0.1 Experiment

A process or action that results in an outcome.

**Examples:**

- Tossing a coin.
- Rolling a die.

#### 1.0.2 Outcome

A single possible result of an experiment.

**Examples:**

- Tossing a coin: Head (H) or Tail (T).
- Rolling a die: 1, 2, 3, 4, 5, 6.

#### 1.0.3 Trial

A single performance of a random experiment.

**Examples:**

- One attempt of rolling a die.
- One toss of a coin.

#### 1.0.4 Sample Space

The set of all possible outcomes of a random experiment.

**Examples:**

- Tossing a coin:  $S = \{H, T\}$ .
- Rolling a die:  $S = \{1, 2, 3, 4, 5, 6\}$ .
- Tossing 2 coins:  $S = \{HH, HT, TH, TT\}$ .
- Rolling 2 dice:  $S = \{(i, j) : i, j = 1, \dots, 6\}$  (36 outcomes).

#### 1.0.5 Event

An event is a subset of the sample space (one or more outcomes).

**Example:** For rolling a die:

$$\begin{aligned} S &= \{1, 2, 3, 4, 5, 6\}, \\ A_{\text{even}} &= \{2, 4, 6\}, \\ A_{\text{odd}} &= \{1, 3, 5\}. \end{aligned}$$

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### 1.0.6 Certain Event

If an event always occurs, its probability is 1.

**Example:** The sun rises in the East.

### 1.0.7 Impossible Event

If an event can never occur, its probability is 0.

**Example:** The sun rising in the West.

### 1.0.8 Elementary Event

A single outcome of a random experiment.

**Example:** Getting Head in one coin toss.

### 1.0.9 Null Event

If two events have no common elements, their intersection is the null set  $\emptyset$ .

**Example:**

$$\begin{aligned}A_{\text{even}} &= \{2, 4, 6\}, \\A_{\text{odd}} &= \{1, 3, 5\}, \\A_{\text{even}} \cap A_{\text{odd}} &= \emptyset.\end{aligned}$$

### 1.0.10 Mutually Exclusive Events

Two events are mutually exclusive if they cannot occur together.

**Example:** In rolling a die,

$$A_{\text{even}} = \{2, 4, 6\}, \quad A_4 = \{1, 2, 3\},$$

then  $A_{\text{even}}$  and  $A_4$  are not mutually exclusive (since they share 2).

### 1.0.11 Equally Likely Events

Events are equally likely if they have the same probability of occurrence.

**Example:** In rolling a die,

$$P(A_{\text{even}}) = \frac{3}{6} = \frac{1}{2}, \quad P(A_{\text{odd}}) = \frac{3}{6} = \frac{1}{2}.$$

### 1.0.12 Exhaustive Events

The set of all possible outcomes of a random experiment.

**Example:** Rolling a die  $\Rightarrow S = \{1, 2, 3, 4, 5, 6\}$ .

## 2 Union, Intersection and Complement of Events

If an experiment is repeated  $n$  times, and event  $A$  occurs  $n(A)$  times, then

$$P(A) = \lim_{n \rightarrow \infty} \frac{n(A)}{n}, \quad 0 \leq P(A) \leq 1. \quad (1)$$

**Events in Repeated Trials:** Suppose we conduct  $n$  independent trials of an experiment:

- 
- Event  $A$  occurs in  $n(A)$  trials.

$$P(A) = \frac{n(A)}{n}$$

- Event  $B$  occurs in  $n(B)$  trials.

$$P(B) = \frac{n(B)}{n}$$

## 2.1 Union of Events

The event  $A \cup B$  (“ $A$  or  $B$ ”) occurs whenever  $A$  happens,  $B$  happens, or both happen.

**Example:**

$$A_{\text{even}} = \{2, 4, 6\}, \quad A_5 = \{1, 2, 3, 4\}, \\ A_{\text{even}} \cup A_5 = \{1, 2, 3, 4, 6\}.$$

From set theory:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In terms of Probability,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

## 2.2 Intersection of Events

The event  $A \cap B$  means both  $A$  and  $B$  occur in the same trial.

**Example:**

$$A_{\text{even}} = \{2, 4, 6\}, \quad A_5 = \{1, 2, 3, 4\}, \\ A_{\text{even}} \cap A_5 = \{2, 4\}.$$

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

In terms of probability:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

## Special Cases

1. **Mutually Exclusive Events:** If  $A$  and  $B$  cannot occur together,

$$P(A \cap B) = 0$$

Hence,

$$P(A \cup B) = P(A) + P(B)$$

2. **Independent Events:** If  $A$  and  $B$  are independent,

$$P(A \cap B) = P(A) \cdot P(B)$$

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## Example

Let  $A$  = event of drawing a King, and  $B$  = event of drawing a Red card.

$$P(A) = \frac{4}{52}, \quad P(B) = \frac{26}{52}, \quad P(A \cap B) = \frac{2}{52}$$

Therefore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$$

Interpretation: Probability of getting either a King or a Red card is  $7/13$ .

## 2.3 Complement of Event

$A^c$  or  $\bar{A}$ : Event containing all outcomes in  $S$  that are not in  $A$ .

**Example:** In rolling a die,

$$A = \{1, 2, 3, 4\}, \quad A^c = \{5, 6\}.$$

In the previous example of Section 2.2,

$$P(A) = \frac{4}{52}$$

Therefore,

$$P(\bar{A}) = 1 - \frac{4}{52} = \frac{48}{52}$$

i.e. the probability of not getting a King is

$$\frac{48}{52}$$

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### 3 Probability Measure and Theorems

**Theorem 1.** In a sample space  $S$ , if  $\bar{A}$  denotes the complement of  $A$ , then

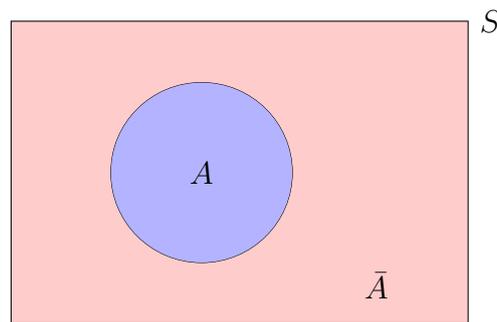
$$P(A) = 1 - P(\bar{A}).$$

*Proof.* The sample space  $S$  can be divided into two mutually exclusive events  $A$  and  $\bar{A}$ . Thus,

$$\begin{aligned} P(S) &= 1, \\ P(A) + P(\bar{A}) &= 1. \end{aligned}$$

Hence,

$$P(A) = 1 - P(\bar{A}).$$



**Theorem 2.** If  $\emptyset$  is an empty set, then  $P(\emptyset) = 0$ .

*Proof.* Let  $A$  be any set such that  $A$  and  $\emptyset$  are mutually exclusive, i.e.,

$$A \cup \emptyset = A.$$

Therefore,

$$P(A \cup \emptyset) = P(A) \implies P(A) + P(\emptyset) = P(A).$$

Thus,  $P(\emptyset) = 0$ .

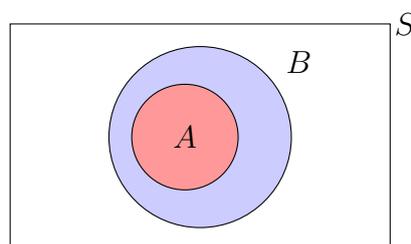
**Theorem 3.** If  $A \subseteq B$ , then  $P(A) \leq P(B)$ .

*Proof.* If  $A \subseteq B$ , then  $B$  can be divided into two mutually exclusive events  $A$  and  $(B - A)$ . Thus,

$$P(B) = P(A) + P(B - A).$$

Since  $P(B - A) \geq 0$  (Probability cannot be negative),

$$P(A) \leq P(B).$$



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**Theorem 4.** If  $A$  and  $B$  are two events, then

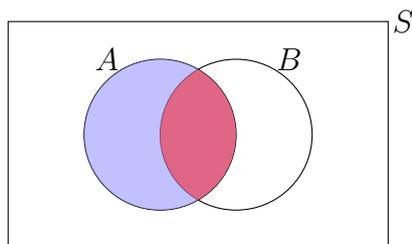
$$P(A - B) = P(A) - P(A \cap B).$$

*Proof.* The event  $A$  can be divided into two mutually exclusive events  $(A - B)$  and  $(A \cap B)$ . Thus,

$$P(A) = P(A - B) + P(A \cap B).$$

Hence,

$$P(A - B) = P(A) - P(A \cap B).$$



**Theorem 5.** If  $A$  and  $B$  are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

*Proof.* The event  $A \cup B$  can be divided into two mutually exclusive events  $(A - B)$  and  $B$ . Thus,

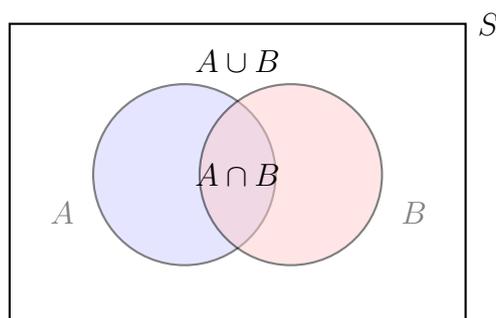
$$P(A \cup B) = P(A - B) + P(B).$$

Using Theorem 4,

$$P(A - B) = P(A) - P(A \cap B).$$

Therefore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



**Special Case:** If  $A \cap B = \emptyset$ , then

$$P(A \cup B) = P(A) + P(B).$$

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## 4 Expectation and Variance

### 4.1 Expectation (Mean)

The expectation (or mean) of a random variable  $X$ , denoted as  $\mathbb{E}[X]$ , is defined as

$$\mathbb{E}[X] = \sum_i x_i P(X = x_i) \quad (\text{discrete case}),$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \quad (\text{continuous case}),$$

where  $f_X(x)$  is the probability density function of  $X$ .

**Ex.** Let  $X$  be the outcome of a fair die roll. Then

$$\mathbb{E}[X] = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = \frac{21}{6} = 3.5.$$

### 4.2 Variance

The variance of a random variable  $X$ , denoted as  $\text{Var}(X)$ , is defined as

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

**Ex.** Let  $X$  be the outcome of a fair die roll. Then

$$\mathbb{E}[X] = 3.5, \quad \mathbb{E}[X^2] = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} = \frac{91}{6}.$$

Thus,

$$\text{Var}(X) = \frac{91}{6} - (3.5)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \approx 2.92.$$

### 4.3 Standard Deviation

The standard deviation of  $X$ , denoted  $\sigma$ , is the square root of variance:

$$\sigma = \sqrt{\text{Var}(X)}.$$

**Ex.** For the fair die roll, we have

$$\sigma = \sqrt{\frac{35}{12}} \approx 1.71.$$

**Interpretation:** The variance measures how far, on average, the outcomes of a random variable deviate from the mean. For a fair die, the expected value is 3.5, but the outcomes vary around this mean.

The variance 2.92 tells us the average squared deviation from the mean, while the standard deviation 1.71 gives a more intuitive measure in the same units as the die outcome.

This means that typical die rolls deviate from the mean (3.5) by about 1.7 units.

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## 5 Worked Examples

**Ex.** If two coins are tossed simultaneously, determine the probability of obtaining exactly two heads.

**Solution:** Number of sample points =  $2 \times 2 = 4$

$$S = \{(T, T), (T, H), (H, T), (H, H)\}$$

The favorable case for exactly two heads is  $(H, H)$ , only 1 outcome.

$$P(\text{two heads}) = \frac{1}{4}.$$

**Ex.** A box contains 3 white, 4 red, and 5 black balls. A ball is drawn at random. Find the probability of: (i) Red (ii) Not black (iii) Black or White.

**Solution:** Total =  $3 + 4 + 5 = 12$ .

(i)  $P(\text{Red}) = \frac{4}{12} = \frac{1}{3}$ .

(ii)  $P(\text{Black}) = \frac{5}{12}$ , so

$$P(\text{Not Black}) = 1 - P(\text{Black}) = 1 - \frac{5}{12} = \frac{7}{12}.$$

(iii)  $P(\text{White or Black}) = P(W) + P(B) = \frac{3}{12} + \frac{5}{12} = \frac{2}{3}$ .

**Ex.** A bag contains 12 balls numbered from 1 to 12. A ball is taken at random. What is the probability that the ball has a number multiple of either 2 or 3?

**Solution:** Let

$$A = \{\text{multiples of 2}\} = \{2, 4, 6, 8, 10, 12\}, \quad P(A) = \frac{6}{12} = \frac{1}{2}.$$

Similarly,

$$B = \{\text{multiples of 3}\} = \{3, 6, 9, 12\}, \quad P(B) = \frac{4}{12} = \frac{1}{3}.$$

The intersection is

$$A \cap B = \{6, 12\}, \quad P(A \cap B) = \frac{2}{12} = \frac{1}{6}.$$

Finally,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} \\ &= \frac{2}{3}. \end{aligned}$$

**Ex.** A coin is tossed four times in succession. Determine the probability of obtaining exactly two heads. (try yourself)

**Ex.** A die is tossed. Find the probability of events  $A = \{\text{odd number}\}$ ,  $B = \{\text{number larger than 3}\}$ ,  $A \cup B$  and  $A \cap B$ .

**Solution:**  $S = \{1, 2, 3, 4, 5, 6\}$ ,  $A = \{1, 3, 5\}$ ,  $B = \{4, 5, 6\}$ . So:

$$P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cup B) = \frac{5}{6}, \quad P(A \cap B) = \frac{1}{6}.$$

Verification:  $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6}$ .

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**Ex.** An experiment consists of rolling a single die. Two events are defined as  $A = \{a \text{ 6 shows up}\}$  and  $B = \{a \text{ 2 or 5 shows up}\}$ . (i) Find  $P(A)$  and  $P(B)$ . (ii)  $P(C) = 1 - P(A) - P(B)$ .

**Solution:**  $P(A) = \frac{1}{6}$ .  $P(B) = \frac{2}{6} = \frac{1}{3}$ .

$$P(C) = 1 - P(A) - P(B) = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}.$$

**Ex.** A pair of dice are thrown. Person A wins if the sum is 6 or less and one die shows 4. Person B wins if the sum is 5 or more and one die shows 4. Find: (a)  $P(A)$ , (b)  $P(B)$ , (c)  $P(A \cap B)$ .

**Solution:** Total number of outcomes = 36.

(a) Favorable outcomes for A:

$$\{(2, 4), (1, 4), (4, 2), (4, 1)\} \Rightarrow 4 \text{ outcomes.}$$

Hence,

$$P(A) = \frac{4}{36}.$$

(b) Favorable outcomes for B:

$$\{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (1, 4), (2, 4), (3, 4), (5, 4), (6, 4)\} \Rightarrow 11 \text{ outcomes.}$$

Hence,

$$P(B) = \frac{11}{36}.$$

(c) Intersection:

$$A \cap B = \{(4, 1), (4, 2), (1, 4), (2, 4)\} \Rightarrow 4 \text{ outcomes.}$$

Therefore,

$$P(A \cap B) = \frac{4}{36}.$$

**Ex.** When three dice are thrown, what is the probability that the sum of numbers on the three faces is less than 16?

**Solution:**

The total number of possible outcomes is

$$6^3 = 216.$$

We need to find

$$P(\text{sum} < 16) = 1 - P(\text{sum} \geq 16).$$

Now, the possible sums  $\geq 16$  are 16, 17, 18.

Number of outcomes with sum 16 = 6

Number of outcomes with sum 17 = 3

Number of outcomes with sum 18 = 1

Thus,

$$\text{Total outcomes with sum } \geq 16 = 6 + 3 + 1 = 10.$$

$$P(\text{sum} \geq 16) = \frac{10}{216}.$$

Therefore,

$$P(\text{sum} < 16) = 1 - \frac{10}{216} = \frac{206}{216}.$$

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**Ex.** Two dice are thrown. Determine: (i)  $P(\text{sum} = 7)$ , (ii)  $P(\text{sum} = 10 \text{ or } 11)$ , (iii)  $P(8 \leq \text{sum} \leq 11)$ , (iv)  $P(\text{sum} > 10)$ .

**Solution:**

The total number of possible outcomes is

$$6 \times 6 = 36.$$

(i) For sum = 7: Possible outcomes = 6

$$P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}.$$

(ii) For sum = 10 or 11: - sum = 10  $\Rightarrow$  3 outcomes - sum = 11  $\Rightarrow$  2 outcomes

$$P(\text{sum} = 10 \text{ or } 11) = \frac{3+2}{36} = \frac{5}{36}.$$

(iii) For  $8 \leq \text{sum} \leq 11$ : - sum = 8  $\Rightarrow$  5 outcomes - sum = 9  $\Rightarrow$  4 outcomes -  
sum = 10  $\Rightarrow$  3 outcomes - sum = 11  $\Rightarrow$  2 outcomes

$$P(8 \leq \text{sum} \leq 11) = \frac{5+4+3+2}{36} = \frac{14}{36} = \frac{7}{18}.$$

(iv) For sum > 10: - sum = 11  $\Rightarrow$  2 outcomes - sum = 12  $\Rightarrow$  1 outcome

$$P(\text{sum} > 10) = \frac{2+1}{36} = \frac{3}{36} = \frac{1}{12}.$$

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## 6 Conditional, Joint Probabilities and Independent Events

### 6.1 Conditional Probability

Let  $A$  and  $B$  be two events of a random experiment. The conditional probability is defined as

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0 \quad (1.4)$$

Here:

- $P(A)$  is called the **elementary probability**.
- $P(A \cap B)$  is the **joint probability**.
- $P(B|A)$  is the **conditional probability**, i.e., the probability of  $B$  given that event  $A$  has already occurred.

### 6.2 Joint Probability

The joint probability of two events may be expressed as the product of the conditional probability of one event given the other, and the elementary probability of the other:

$$P(A \cap B) = P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$$

This is called the **chain rule** or **multiplication rule**.

If the occurrence of event  $A$  does not affect the occurrence of event  $B$ , then

$$P(B|A) = P(B) \quad \Rightarrow \quad P(A \cap B) = P(A) \cdot P(B) \quad (1.5)$$

In this case,  $A$  and  $B$  are said to be **independent events**.

If  $P(A \cap B) = \emptyset$  (null event), then  $A$  and  $B$  are called **mutually exclusive events**.

### Question 12

In a box there are 100 resistors with resistances and tolerances as given in Table 1.

Resistance ( $\Omega$ )	5%	10%	Total
22	10	14	24
47	28	16	44
100	24	8	32
Total	62	38	100

Table 1: Resistors with resistance and tolerance.

Define the events:

$$\begin{aligned} A &= \{\text{drawing a } 47 \Omega \text{ resistor}\}, \\ B &= \{\text{drawing a } 5\% \text{ tolerance resistor}\}, \\ C &= \{\text{drawing a } 100 \Omega \text{ resistor}\} \end{aligned}$$

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## Solution

1. Elementary probabilities:

$$P(A) = \frac{44}{100}, \quad P(B) = \frac{62}{100}, \quad P(C) = \frac{32}{100}$$

2. Joint probabilities:

$$P(A \cap B) = \frac{28}{100}, \quad P(B \cap C) = \frac{24}{100}, \quad P(A \cap C) = 0$$

3. Conditional probabilities:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{28/100}{62/100} = \frac{28}{62} = \frac{14}{31}$$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0}{32/100} = 0$$

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{24/100}{32/100} = \frac{24}{32} = \frac{3}{4}$$

4. Independence check: Compare  $P(A \cap B)$  and  $P(A) \cdot P(B)$ :

$$P(A \cap B) = \frac{28}{100} = \frac{7}{25}$$

$$P(A) \cdot P(B) = \frac{44}{100} \cdot \frac{62}{100} = \frac{2728}{10000} = \frac{682}{2500} = \frac{341}{1250}$$

Since

$$P(A \cap B) \neq P(A) \cdot P(B),$$

events  $A$  and  $B$  are **dependent**.

Similarly,  $P(B \cap C) \neq P(B) \cdot P(C)$  and  $P(A \cap C) \neq P(A) \cdot P(C)$ , hence  $A, B, C$  are not independent.

## 7 Properties of Conditional Probability

1. For any two events  $A$  and  $B$  in the sample space. If  $B \subseteq A$ , then

$$P(A|B) = 1$$

**Proof:** From Fig. 1.4(i),

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

2. If  $B \subseteq A$ , then

$$P(B|A) = \frac{P(B)}{P(A)}$$

**Proof:**

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}$$

3.  $P(A|B) \geq 0$ , i.e., conditional probability is non-negative. **Proof:**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Since  $P(A \cap B) \geq 0$  and  $P(B) \neq 0$ ,

$$P(A|B) \geq 0$$

4. If two events  $A$  and  $B$  are in the sample space  $S$ , then:

- $P(S|A) = P(S|B) = 1$  **Proof:** From Fig. 1.4(ii) and (iii),

$$P(S|A) = \frac{P(S \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1, \quad P(A) \neq 0$$

Similarly,

$$P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1, \quad P(B) \neq 0$$

- $P(A|S) = P(A)$  and  $P(B|S) = P(B)$  **Proof:**

$$P(A|S) = \frac{P(A \cap S)}{P(S)} = \frac{P(A)}{1} = P(A)$$

Similarly,

$$P(B|S) = \frac{P(B \cap S)}{P(S)} = \frac{P(B)}{1} = P(B)$$

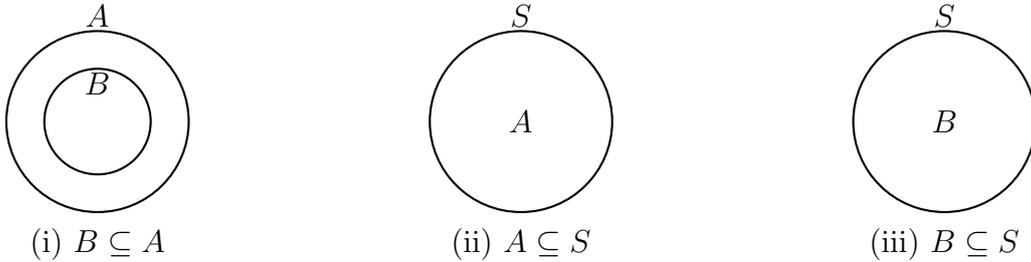


Figure 1: Venn diagrams illustrating conditional probability cases.

## 8 Joint Properties and Independent Events

### 8.1 Joint Properties

$$P(A \cap B) = P(A) \cdot P(B|A)$$

### 8.2 Independent Events

If

$$P(B|A) = P(B),$$

then

$$P(A \cap B) = P(A) \cdot P(B)$$

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If

$$P(A \cap B) = 0,$$

then  $A$  and  $B$  are called **mutually exclusive events**.

For  $n$  events  $A_1, A_2, \dots, A_n$  in a random experiment, if the events are independent, then the probability multiplication theorem holds:

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \dots \cdot P(A_n)$$

### Theorem 1.4.6

If events  $A$  and  $B$  are independent, then the following pairs are also independent:

$$A \text{ and } \bar{B}, \quad \bar{A} \text{ and } B, \quad \bar{A} \text{ and } \bar{B}$$

### Proof

1. Independence of  $A$  and  $\bar{B}$ :

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

Since  $A$  and  $B$  are independent,  $P(A \cap B) = P(A)P(B)$ . Thus:

$$P(A \cap \bar{B}) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(\bar{B})$$

2. Independence of  $\bar{A}$  and  $B$ :

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Again,  $P(A \cap B) = P(A)P(B)$ , so:

$$P(\bar{A} \cap B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(\bar{A})P(B)$$

3. Independence of  $\bar{A}$  and  $\bar{B}$ :

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since  $P(A \cap B) = P(A)P(B)$ , we have:

$$\begin{aligned} P(\bar{A} \cap \bar{B}) &= 1 - (P(A) + P(B) - P(A)P(B)) \\ &= (1 - P(A))(1 - P(B)) = P(\bar{A})P(\bar{B}) \end{aligned}$$

Hence, if  $A$  and  $B$  are independent, then  $A$  and  $\bar{B}$ ,  $\bar{A}$  and  $B$ , and  $\bar{A}$  and  $\bar{B}$  are also independent.

$$\begin{aligned} P(A \cap \bar{B}) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \end{aligned}$$

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## 9 Law of Total Probability

Let  $B_1, B_2, \dots, B_n$  be a partition of the sample space  $S$ , i.e.,

- $B_1 \cup B_2 \cup \dots \cup B_n = S$
- $B_i \cap B_j = \emptyset \quad (i \neq j)$

Then, for any event  $A$ ,

$$P(A) = \sum_{i=1}^n P(A \cap B_i).$$

Using conditional probability,

$$P(A \cap B_i) = P(A|B_i) P(B_i).$$

Hence,

$$P(A) = \sum_{i=1}^n P(A|B_i) P(B_i)$$

This is known as the **Law of Total Probability (LTP)**.

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### Example 1: Medical Test

A population is divided into two groups:

- $M_1$ : Male,  $P(M_1) = 0.4$
- $M_2$ : Female,  $P(M_2) = 0.6$

Disease probability:

$$P(D|M_1) = 0.01, \quad P(D|M_2) = 0.02.$$

Total probability:

$$\begin{aligned} P(D) &= P(M_1)P(D|M_1) + P(M_2)P(D|M_2) \\ &= (0.4)(0.01) + (0.6)(0.02) = 0.016. \end{aligned}$$

So, the probability that a random person has the disease is 1.6%.

### Example 2: Rolling a Die

Two dice:

- $D_1$ : Fair die,  $P(D_1) = 0.5$
- $D_2$ : Loaded die,  $P(D_2) = 0.5$

Probabilities of rolling a six:

$$P(6|D_1) = \frac{1}{6}, \quad P(6|D_2) = \frac{1}{2}.$$

By total probability:

$$P(6) = (0.5) \left(\frac{1}{6}\right) + (0.5) \left(\frac{1}{2}\right) = \frac{1}{3}.$$

So, probability of rolling a 6 is 0.3333.

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### Example 3: Defective Bulb

Three factories:

- $F_1$ : 20% of bulbs, defect rate = 2%
- $F_2$ : 50% of bulbs, defect rate = 3%
- $F_3$ : 30% of bulbs, defect rate = 4%

Total probability of defect:

$$\begin{aligned}P(D) &= (0.2)(0.02) + (0.5)(0.03) + (0.3)(0.04) \\ &= 0.004 + 0.015 + 0.012 = 0.031.\end{aligned}$$

So, probability of a defective bulb = 3.1%.

### Example 4:

A factory has three machines  $M_1, M_2, M_3$  producing 40%, 35%, and 25% of items, respectively. The probability of producing a defective item is:

$$P(D|M_1) = 0.01, \quad P(D|M_2) = 0.02, \quad P(D|M_3) = 0.03.$$

Find the probability that a randomly chosen item is defective.

$$P(D) = P(D|M_1)P(M_1) + P(D|M_2)P(M_2) + P(D|M_3)P(M_3).$$

$$P(D) = (0.01)(0.40) + (0.02)(0.35) + (0.03)(0.25) = 0.0185.$$

Thus, the overall defective probability is **1.85%**.

## 10 Bayes' Theorem

**Statement:** For events  $B_1, B_2, \dots, B_n$  forming a partition of  $S$ , and any event  $A$  with  $P(A) > 0$ ,

$$P(B_j|A) = \frac{P(A|B_j) P(B_j)}{\sum_{i=1}^n P(A|B_i) P(B_i)}.$$

This is known as **Bayes' Theorem**.

**Bayes' Theorem** allows us to update prior probabilities  $P(B_i)$  in light of new evidence  $A$ .

Thus, if an item is defective, there is about a **37.8% chance** it came from  $M_2$ .

### Example: Defective Bulb Source

Using Example 3, suppose a defective bulb is found. What is the probability it came from  $F_3$ ?

$$\begin{aligned}P(F_3|D) &= \frac{P(F_3)P(D|F_3)}{P(D)} \\ &= \frac{(0.3)(0.04)}{0.031} = \frac{0.012}{0.031} \approx 0.387.\end{aligned}$$

So, there is about 38.7% chance the defective bulb came from Factory  $F_3$ .

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## Example: Medical Diagnosis

Suppose a disease test has:

- Sensitivity:  $P(\text{Positive}|D) = 0.95$  (true positive rate)
- Specificity:  $P(\text{Negative}|D^c) = 0.90$  (true negative rate)

Prevalence of the disease in the population:

$$P(D) = 0.01, \quad P(D^c) = 0.99.$$

We want:  $P(D|\text{Positive})$ .

By Bayes' Theorem:

$$\begin{aligned} P(D|\text{Positive}) &= \frac{P(D)P(\text{Positive}|D)}{P(D)P(\text{Positive}|D) + P(D^c)P(\text{Positive}|D^c)} \\ &= \frac{(0.01)(0.95)}{(0.01)(0.95) + (0.99)(0.10)} \\ &= \frac{0.0095}{0.0095 + 0.099} = \frac{0.0095}{0.1085} \approx 0.0876. \end{aligned}$$

So, even with a positive test, the probability of actually having the disease is only about 8.8% due to low prevalence.

## Example: Email Spam Filter

A spam filter detects spam with the following probabilities:

- $P(\text{Spam}) = 0.2$
- $P(\text{Not Spam}) = 0.8$
- $P(\text{Word "Win"}|\text{Spam}) = 0.6$
- $P(\text{Word "Win"}|\text{Not Spam}) = 0.05$

We want  $P(\text{Spam}|\text{"Win"})$ .

By Bayes' Theorem:

$$\begin{aligned} P(\text{Spam}|\text{Win}) &= \frac{P(\text{Spam})P(\text{Win}|\text{Spam})}{P(\text{Spam})P(\text{Win}|\text{Spam}) + P(\text{Not Spam})P(\text{Win}|\text{Not Spam})} \\ &= \frac{(0.2)(0.6)}{(0.2)(0.6) + (0.8)(0.05)} \\ &= \frac{0.12}{0.12 + 0.04} = \frac{0.12}{0.16} = 0.75. \end{aligned}$$

So, if the word "Win" appears, the probability the email is spam is 75%.

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## Example: Student Choosing a Subject

At a college, students must choose one of three electives: **E1**, **E2**, or **E3**.

- 40% of students choose E1.
- 35% of students choose E2.
- 25% of students choose E3.

The probability of passing the final exam given the elective is:

$$P(\text{Pass}|E1) = 0.8, \quad P(\text{Pass}|E2) = 0.6, \quad P(\text{Pass}|E3) = 0.9$$

## Solution

The events  $\{E1, E2, E3\}$  form a partition of the sample space.

Using the law of total probability, the overall probability that a random student passes is

$$P(\text{Pass}) = P(E1)P(\text{Pass}|E1) + P(E2)P(\text{Pass}|E2) + P(E3)P(\text{Pass}|E3)$$

$$P(\text{Pass}) = (0.40)(0.8) + (0.35)(0.6) + (0.25)(0.9)$$

$$P(\text{Pass}) = 0.32 + 0.21 + 0.225 = 0.755$$

Therefore, the overall probability that a student passes the exam is

$$\boxed{0.755 \text{ or } 75.5\%}$$