Transformation matrix in 3D.

$$A = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & h & i & r \\ l & m & n & s \end{bmatrix} = \begin{bmatrix} T & K \\ \tau & \theta \end{bmatrix}$$ where

$T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ produces linear transformations : scaling, shearing, reflection & rotation.

$K = [p \; q \; r]^T$, produces translation

$\tau = [l \; m \; n]^T$, yields perspective transformation

$\theta = s$, is responsible for uniform scaling.

Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad x' = x \cdot S_x, \; y' = y \cdot S_y, \; z' = z \cdot S_z$$

Shear

$$\begin{bmatrix} 1 & 0 & Sh_x & 0 \\ 0 & 1 & Sh_y & 0 \\ 0 & Sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Origin is unaffected by scaling & shear. It can be translated.

Scaling w.r.t to fixed pt. (x_f, y_f, z_f)

$$T(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f)$$

$$= \begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f \\ 0 & S_y & 0 & (1-S_y)y_f \\ 0 & 0 & S_z & (1-S_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse scaling is obtained by taking reciprocals of scale parameters.

Reflection

$$T_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

produce reflection about; xy plane

$$T_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

refl^n about; yz plane

$$T_{zx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reflection about : zx plane

Rotation

$x \to y \to z \to x.$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = R_z(\theta) \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta.$

$z' = z.$

Reflection

$$T_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

produce reflection about ; xy plane

$$T_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

refl^n about ; yz plane

$$T_{zx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reflection about : zx plane

Rotation

$x \to y \to z \to x$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = R_z(\theta) \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$z' = z$$

Rotation along x-axis

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

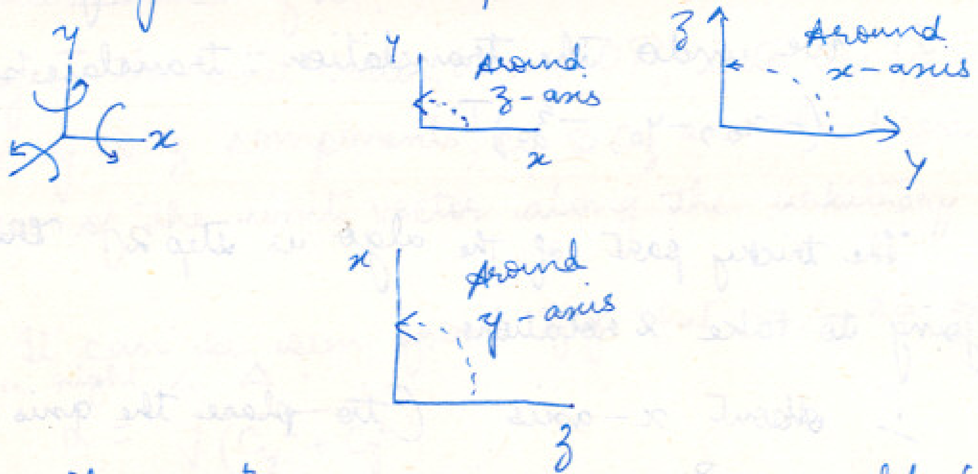
$y' = y\cos\theta - z\sin\theta$
$z' = y\sin\theta + z\cos\theta$
$x' = x$

Rotation along y-axis.

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$z' = z\cos\theta - x\sin\theta$
$x' = z\sin\theta + x\cos\theta$
$y' = y$.

Why is the sign reversed in $R_y(\theta)$.



if we put $\cos\theta \quad 0 \quad -\sin\theta$ we would have forced the rotation from x to z & not z to x that is why the 3rd row has a -ve sign.

$$\begin{array}{c c} \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \leftarrow x \\ \leftarrow y \\ \leftarrow z \\ \end{array} \end{array}$$

Rot^n about an arbitrary axis in space

Assume, we want to perform a rotation by θ degrees, about an axis in space passing through the point (x_0, y_0, z_0) with direction cosines (c_x, c_y, c_z).

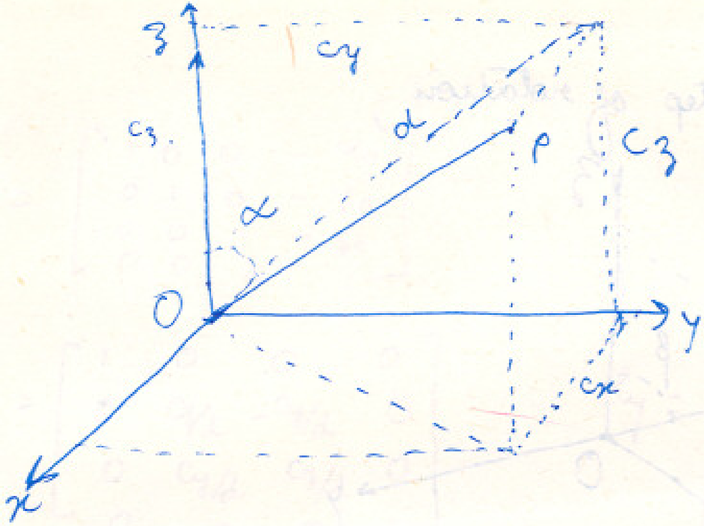
1. First of all translate by $|T| = -(x_0, y_0, z_0)^T$
2. Next, we rotate the axis into one of the principle axes, let's pick, z $(|R_x|, |R_y|)$.
3. We rotate next by θ degrees in z $\left(|R_z(\theta)|\right)$.
4. Then we undo the rotations to align the axis.
5. We undo the translation: translate by $(-x_0, -y_0, -z_0)^T$

The tricky part of the algo is step 2. This is going to take 2 rotations.

i. About x-axis (to place the axis in the xz plane)

&

ii. About y-axis (to place the result coincident with z-axis).



First step of rotation

OP is a line about which rot^n is to take place

Rotation about x by α

Determination of α?

$\boxed{C_z, C_y \text{ \& } d \text{ is a right } \angle\ \Delta.}$

Project the unit vector, along OP, into the yz plane, The y & z components, c_y & C_z are the direction cosines of the unit vector along the arbitrary axis.

It can be seen from fig that C_z, C_y & d form a right $\angle$ Δ. $\therefore$

$$d = \sqrt{(C_y^2 + C_z^2)}$$

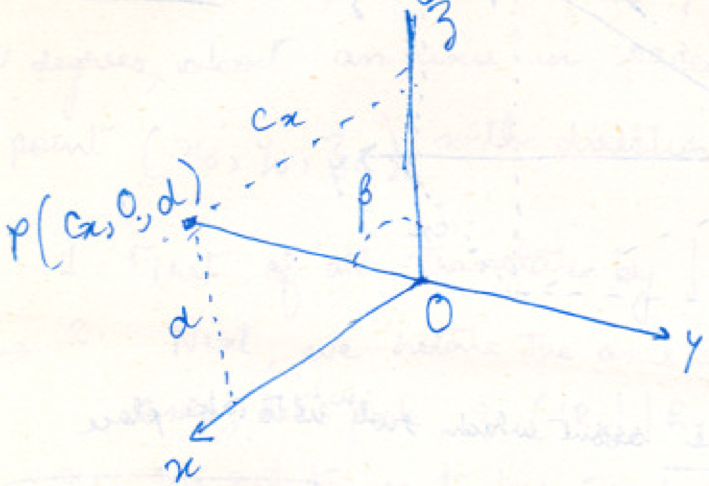
~~Cos o~~ $\cos\alpha = C_z/d$

$$\sin\alpha = C_y/d$$

$$\alpha = \sin^{-1}\left[\frac{C_y}{\sqrt{C_y^2 + C_z^2}}\right]$$

After the 1st rotation OP would be lying along the zx plane.

after 1st step of rotation,



$\sqrt{c^2+d^2} = 1$

$\therefore$ OP is a unit vector,

Rot^n about y by β :

- Determine the angle β to rotate the result into the z-axis
- The x component is c_x & z component is d.

$\cos\beta = d = d / (\text{length of unit vector})$

$\sin\beta = c_x = c_x / (\text{length of unit vector})$. $-c_x$

Final transformation for 3D rotation, about an arbitrary axis :

$$M = |T|\,|R_x|\,|R_y|\,|R_z|\,|R_y|^{-1}\,|R_x|^{-1}\,|T|^{-1}$$

where,

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_z/d & -c_y/d & 0 \\ 0 & c_y/d & c_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_y = \begin{bmatrix} d & 0 & -c_x & 0 \\ 0 & 1 & 0 & 0 \\ c_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since rot^{ns} are orthogonal matrices ∴ Inverse rot^n are given by transpose matrices resp.

$$M = |T|\,|R_x|\,|R_y|\,|R_z|\,|R_y|^{-1}\,|R_x|^{-1}\,|T|^{-1}$$

$$= [T\,R_x\,R_y]\,[R_z]\,[T\,R_x\,R_y]^{-1}$$

$$= C\,[R_z]\,C^{-1}$$

A special case of 3D rotation:

Rotation about an axis parallel to a coordinate axis (say parallel to x-axis):

$$M_x = |T|\,|R_x|\,|T|^{-1}$$

If you are given 2 pts instead (on the axis of rotation) you can calculate the direction cosines of the axis as follows:

$$V = \left| (x_1 - x_0)\ (y_1 - y_0)\ (z_1 - z_0) \right|^T$$

$$C_x = (x_1 - x_0) / |V|$$

$$C_y = (y_1 - y_0) / |V|$$

$$C_z = (z_1 - z_0) / |V|$$

$$[c_x\ c_y\ c_z] = \frac{(x_1 - x_0)(y_1 - y_0)(z_1 - z_0)}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}$$

$$c_x = \frac{x_1 - x_0}{|V|}$$

$|V| =$

where $|V|$ is the length of the vector V.

The unit vector u (or) is defined along the rot^n axis as,

$$u = \frac{V}{|V|} = (c_x, c_y, c_z).$$

<u>Reflection through an arbitrary plane</u>

method is similar to rotation about an arbitrary axis.

$$M = |T|\,|R_x|\,|R_y|\,|R_{fl}|\,|R_y|^{-1}\,|R_x|^{-1}\,|T|^{-1}$$

T does the job of translating the origin to the plane.

R_x & R_y will rotate the vector normal to the reflection plane (at the origin), until it coincides with $+z$ axis.

R_{fl} is the reflection matrix about x-y plane or $z = 0$ plane.